Matix, eigenvalues & eigenvectors.

Diagonalizability of Matrices

Exercise III pg (4-73)

1) Diagonalise the matrix $\begin{bmatrix} -3 & 2+2i \\ 2-2i & 4 \end{bmatrix}$

For diagonalizing, we must first find the eigenvalues.

The characteristic equation of the above matrix is as follows:-

$$\lambda^2 - \lambda + (-20) = 0$$

$\cancel{\lambda^2 - \lambda - 20 = 0}$

$\cancel{(\lambda+3)(\lambda-4) = 0}$

$\lambda = \cancel{-3, 4}, \cancel{-4}, 5$

$$(\lambda-5)(\lambda+4) = 0$$

$$\lambda = 5, -4$$

$\therefore$ The matrix is diagonalizable, and the diagonal matrix is

$\cancel{\begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix}}$ $\begin{bmatrix} 5 & 0 \\ 0 & -4 \end{bmatrix}$

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2) Reduce the matrix to diagonal form

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

To reduce to diagonal form, we must first find all eigenvalues

Characteristic equation for the above matrix is

$$\lambda^3 - (\text{trace of } \cancel{\text{matrx}}\ \text{matrix})\lambda^2 + (\text{sum of diagonal minors})\lambda - |A| = 0$$

$$\lambda^3 - (9)\lambda^2 + (8+8+8)\lambda - (27+1+1-3-3-3) = 0$$

$$\lambda^3 - 9\lambda^2 + 24\lambda - 20 = 0$$

$$\lambda = 5, 2, 2$$

We can see that we have repeating eigenvalues

Since one eigenvalue is repeated 2 times, the algebraic multiplicity of the matrix is 2.

Now, let's try and find the eigenvectors since we need to find the geometric multiplicity.

For $\lambda = 2$,

The equation $[A - \lambda I]X = 0$

shall become

$\Rightarrow$

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Applying row transformations,

$R_3 - R_1 \;/\; R_2 + R_1$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

We can see that the rank of the matrix is 1, and the number of variables is 3.

$\therefore$ there must exist $3-1=2$ linearly independent solutions.

Assuming two parameters 's' & 't'

putting $x_2 = s$, $x_3 = -t$

$x_1 = +x_2 - x_3$
$x_1 = -s + t$

$$\therefore X = \begin{bmatrix} -s+t \\ -s \\ -t \end{bmatrix} = s \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\therefore X_1 = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} \; \& \; X_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Since geometric multiplicity is also 2.

A.M = G.M $\therefore$ Matrix is diagonalizable.

∴ Diagonal Matrix

$$= \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

3) $$A = \begin{bmatrix} 1 & -2 & 0 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

Is this matrix diagonalizable?

Let's first find out the eigenvalues by writing the characteristic equation :-

$$\Rightarrow \lambda^3 - (6)\lambda^2 + (2+3+4)\lambda - (6-4-4+6) = 0$$

$$\Rightarrow \lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$$

$$\lambda = 4, 1, 1$$

We can see that there is a repeated eigenvalue, which makes the Algebraic Multiplicity = 2

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Let's find the eigenvector corresponding to $\lambda = 1$ for checking it's ~~eigen~~. geometric multiplicity :-

for $\lambda = 1$, $[A - \lambda I]X = 0$ becomes,

$$\begin{bmatrix} 0 & -2 & 0 \\ 1 & 1 & 2 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$R_3 \leftrightarrow R_1$

$$\begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & 2 \\ 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$R_2 - R_1$

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & -1 & 0 \\ 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$R_3 - 2R_2$

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now, looks like the rank of the matrix is 2, and 3 variables present.

$\therefore$ linearly independent solutions = 3-2 = 1.

$\Downarrow$

geometric multiplicity (G·M)

$\therefore$ A·M $\neq$ G·M

2 $\neq$ 1

$\therefore$ Matrix is not diagonalizable.

Function of a square matrix

→ If a matrix A is a singular square matrix, we can find the modal matrix M & diagonal matrix D if it has distinct eigenvalues.

→ We can use these to ~~fing~~ ~~to~~ find functions of A like $A^n, e^A, 4^A$ etc.

(a) Calculation of powers of matrix

We have, ~~$A = M$~~

$$D = M^{-1}AM$$

$$\Rightarrow A = MDM^{-1}$$

for A^n

$$\Rightarrow (MDM^{-1})(MDM^{-1}) \cdots \text{ n times}$$

$$\Rightarrow MD(M^{-1}M)DM^{-1} \cdots$$

$$\Rightarrow MD^2M^{-1} \cdots$$

$$\therefore A^n \Rightarrow MD^nM^{-1}$$

Similarly for e^A & b^A.

(b) If the diagonal mat. of A cannot be found out, we can use the method below.

for a matrix of order 3, the required function can be written as:

$$\phi(A) = \alpha_2 A^2 + \alpha_1 A + \alpha_0 I$$

[α_n is constant]

similaly for order of 2,

$$\phi(A) = \alpha_1 A + \alpha_0 I$$

All notes referred from

Applied Mathematics – 4
~ G.V. Kumbhojkar.