Generalized Calabi-Yau Manifolds and B-Fields

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1 Introduction/Overview

We first recall the definition of a generalized complex structure:

Definition 1.1. Given a smooth manifold M (with $dim(M) = 2m$), along with an indefinite metric on the generalized bundle $T \oplus T^*$ given by $(v + \xi, v + \xi) = -\langle v, \xi \rangle$ a *generalized complex structure* on M is a subbundle $E \subset (T \oplus T^*) \otimes \mathbb{C}$ such that the following hold:

- $E \oplus \bar{E} = (T \oplus T^*) \otimes \mathbb{C}$
- The space of E-sections is closed under the Courant bracket
- E is isotropic

Definition 1.2. A *generalized Calabi-Yau manifold* is a smooth manifold M of dimension $2m$ with a closed form $\varphi \in \Omega^{ev} \otimes \mathbb{C}$ or $\varphi \in \Omega^{od} \otimes \mathbb{C}$ that is a complex pure spinor for the generalized (tangent) bundle $T \oplus T^*$, and such that $\langle \varphi, \bar{\varphi} \rangle \neq 0$ at each point in M.

Some clarifications for the definition:

- Ω^{ev} and Ω^{od} are the spaces of even and odd forms.
- $\langle \varphi, \bar{\varphi} \rangle$ is defined as follows:
  If $\varphi \in \Omega^{ev} \otimes \mathbb{C}$, then

$$\langle \varphi, \bar{\varphi} \rangle = \sum_n (-1)^n \varphi_{2n} \wedge \bar{\varphi}_{2m-2n},$$

and if $\varphi \in \Omega^{od} \otimes \mathbb{C}$, then

$$\langle \varphi, \bar{\varphi} \rangle = \sum_n (-1)^n \varphi_{2n+1} \wedge \bar{\varphi}_{2m-2n-1}.$$

Definition 1.3. Let V be an n-dimensional real vector space. Define

$$S^+ = \Lambda^{ev} V^* \otimes (\Lambda^n V)^{1/2},$$
$$S^- = \Lambda^{od} V^* \otimes (\Lambda^n V)^{1/2}.$$

Given $\varphi \in S^\pm$, the *annihilator* of φ is the vector space

$$E_\varphi = \{v + \xi \in V \oplus V^* | (v + \xi) \cdot \varphi = 0\}.$$

Definition 1.4. A *spinor* is an element of a spin representation of the orthogonal Lie algebra. A *pure spinor* is a spinor φ for which E_φ is maximally isotropic (has dimension equal to that of V).

Clarification:

- $S^\pm$ are irreducible half-spin representations of $V \oplus V^*$. So, elements in them are spinors.

Lemma 1.1. If (M, φ) is a generalized Calabi-Yau manifold, then the annihilator $E_\varphi \subset (T \oplus T^*) \otimes \mathbb{C}$ defines a generalized complex structure on M.

Proof. The annihilator of a pure spinor is isotropic, and E_φ is a pure spinor, so it is isotropic. Since $\langle \varphi, \bar{\varphi} \rangle \neq 0$, we have that $0 = E_\varphi \cap E_{\bar{\varphi}} = E_\varphi \cap \bar{E}_\varphi$. Thus,

$$E_\varphi \oplus \bar{E}_\varphi = (T \oplus T^*) \otimes \mathbb{C}.$$

Finally, we have to show that the sections of E_φ are closed under the Courant bracket. This follows from a bit of algebra, which we omit here. □

One interesting example of a generalized Calabi-Yau structure arises from symplectic geometry, as follows.

Example 1.1. Suppose we have a symplectic manifold M with symplectic form ω. $1 \in \Lambda^0 T^*$ is pure, which implies that $\varphi = \exp(i\omega) \in \Omega^{ev} \otimes \mathbb{C}$ is pure as well. Bilinearity gives us

$$\langle \varphi, \bar{\varphi} \rangle = \frac{(-2i)^m}{m!} \omega^m$$

This is nonvanishing, and $d\omega = 0$, which means $\varphi = \exp(i\omega)$ defines a generalized Calabi-Yau manifold.

2 The B-Field

Suppose we have a real, closed 2-form B, along with a generalized Calabi-Yau manifold (M, φ). From the previous section, we know that the exponentiation $(\exp B)\varphi = (1 + B + \frac{1}{2}B^2 + \ldots) \wedge \varphi$ is pure. We also know that $\exp B$ is real, and acts through the Spin group by

$$\langle \exp B\varphi, \exp B\bar{\varphi} \rangle = \langle \varphi, \bar{\varphi} \rangle \neq 0.$$

This is exciting - it tells us that any generalized Calabi-Yau structure can be transformed by a B-field to another Calabi-Yau structure! In the symplectic case, $\varphi = exp(B + i\omega)$ provides the requisite B-field transformation. Because φ may always be multiplied by a complex constant, we have a particularly large class of generalized Calabi-Yau manifolds $\psi = c\exp(B + i\omega)$, $c \in \mathbb{C}$.